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Determining the Pareto set in a bicriteria two-echelon inventory/distribution system

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This article concerns two-echelon inventory/distribution system, consisting of a warehouse and a retailer. We assume that the demand is deterministic and stockouts are not permitted. Two criteria are considered: to minimize the annual inventory cost and the annual total number of damaged items by improper shipment handling. The problem consists of determining the non-dominated inventory policies in such a way that the trade-off between both criteria is achieved. We present the characterization of the non-dominated optimal solution set and we use this result to correct the solution method previously proposed by other authors for a problem with identical cost structure. An efficient algorithm to calculate the non-dominated solution set is introduced. Computational results on several randomly generated problems are reported.

Keywords: Pareto set; two-echelon inventory/distribution system; level curves; warehouse

2000 Mathematics Subject Classification: 90B05; 90B06; 90C29

1. Introduction

We deal with two-echelon inventory/distribution (I/D) systems, where it is appropriate to coordinate the control of different stock keeping units. We look at the case of an item being stocked at two locations with resupply being made between them.

The retail outlet is replenished from the warehouse which is supplied from an outside supplier. In such a situation, coordinated control makes sense in that, for example, replenishment decisions at the retailer outlet impinge as demand on the warehouse. We consider that the demand at the warehouse is dependent on the demand (and stocking decisions) of the customers. We refer to this as a dependent demand situation in contrast with classical demands for different stock keeping units, which are considered as being independent. At each location, we assume that stockouts are not permitted and a continuous review economic order quantity policy is used.

When a unique criterion is considered, the decision involves the choice of a lot size for each facility (warehouse and retailer), which minimizes the inventory cost, that is, the sum

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of the holding cost plus the ordering cost at both the warehouse and retailer. Determination of the optimal policy for a two-echelon serial I/D system is not obvious, mainly because of the complex interactions between echelons [7]. Considering deterministic demand, however, it is possible to model a multi-echelon system using the concept of echelon stock, first introduced by Clark and Scarf [3]. They defined the echelon stock of echelon j (in a general multi-echelon system) as the number of units in the system that are at, or have passed through, echelon j but have as yet not been specifically committed to outside customers (when backorders are permitted the echelon stock can be negative). With this definition and uniform end-item demand, each echelon stock has a sawtooth pattern with time.

Taking into account the integer-ratio policy proposed by Taha and Skeith [10] (its optimality for two-echelon systems was proved by Crowston *et al.* [4] and Williams [11]), it is simple to compute the average value of an echelon stock and the echelon holding costs. This policy tells us that an optimal set of lot sizes exists such that the lot size at each facility is a positive integer multiple of the lot size at its successor facility. This fact was used by Crowston et al. [4] in the development of a dynamic programming approach for determining optimal lot sizes. Some other interesting models about multi-echelon systems are also studied in Silver and Peterson [8].

Traditional approaches for multi-echelon I/D systems usually have one global objective, cost minimization, typically optimized in an unconstrained manner. However, new approaches in multi-echelon systems considering different objectives have been also developed. In particular, these objectives involved in inventory management concern the reduction of the inventory cost, the minimization of the transportation cost, the reduction of the expected number of stockouts per year (customer service), among others. Accordingly, the goal of these problems consists of determining the set of non-dominated solutions, also known as Pareto-optimal set or efficient solution set, which contains solutions (vectors) where none of the components can be improved without deterioration to at least one of the other components in the objective space.

A significant number of researchers in inventory management have made notable efforts to deal with more than one performance measure or objective. Star and Miller [9] determined a trade-off between two measures: the number of annual orders (i.e. workload) and the average investment in the inventory. They developed the concept of an optimal policy curve, where the points on this curve represent policies between which the decision maker is indifferent. Points off the curve are either infeasible or sub-optimal, but can be improved by moving back to the curve. Gardnet and Dannenbring [5] extended the above concept to a three-dimensional optimal policy surface by adding the performance measure of customer service when they analyse a probabilistic multi-item distribution system. Brown [2] also derived an exchange curve between two performance measures such as workload, investment in inventory or customer service for both deterministic and probabilistic inventory problems. Zeleny [12] discussed Star and Miller's work in the sense that, the optimal policy curve (or surface) is equivalent to non-dominated solutions in Multiple Criteria Decision Making (MCDM). Recently, Puerto and Fernández [6] also analysed some inventory models from the MCDM perspective.

Bookbinder and Chen [1] applied the MCDM methodology to a two-echelon serial inventory/distribution system. They discussed different models with deterministic and probabilistic demand, and they assumed that marginal inventory costs were known. Three non-linear multiobjetive programming models and corresponding solution approaches were presented to obtain non-dominated inventory policies achieving trade-offs among objectives such as customer service, inventory investment and transportation cost. Their results were MCDM generalizations of Brown's exchange curve, Star and Miller's optimal policy curve and Gardne and Dannenbring's optimal policy surface.

In this article, we show a new MCDM approach for determining all the admissible lot sizes for a two-echelon inventory/distribution system considering deterministic demand. This problem can be seen as a two-objective non-linear mixed-integer programming model. The first objective consists of minimizing the annual inventory cost, i.e. the sum of the total holding cost plus ordering costs at both warehouse and retailer. The second objective concerns the minimization of the annual total number of damaged items by improper shipment handling, which is assumed to be dependent on the number of shipments per year and on the order quantity. Thus, as the annual number of shipments increases so does the number of items which could be damaged due to negligence of the personnel handling the items. The minimization of this latter objective is mainly justified when fragile goods are handled. In addition, two constrains are considered: the first concerns the retailer inventory capacity and the capacity of the vehicle for delivery, and the second one is related to the restriction of the integer-ratio policy previously mentioned. We solve the problem exactly by finding the whole set of non-dominated policies by means of an exhaustive case analysis of the model.

Notice that the cost structure of the problem under study is similar to that presented in Bookbinder and Chen [1]. Therefore, as it could be expected, their solution approach should give the non-dominated solution set for our problem as well. However, as we will prove further on, their solution method for the deterministic case is not correct since it provides no good solutions, generating dominated policies.

The rest of the article is organized as follows. In Section 2, we introduce some notation and we state the model. We continue in Section 3, introducing some preliminary results, which simplify the determination of the Pareto solution set. In Section 4, the form of the non-dominated solution set is studied. This form depends on several cases as the results in Section 4 show. In addition, we use our solution method, in Section 5, to show that the approach proposed by Bookbinder and Chen to calculate efficient policies is not correct. Some computational results are discussed in Section 6. We conclude in Section 7 with a summary and a brief discussion of implications of the model.

2. Model formulation and notation

We consider a two-echelon inventory/distribution system where a single item is provided by an outside supplier, stocked at the warehouse and distributed to customers through one retailer.

It is assumed throughout that the demand is known with certainty. Perhaps, this is admittedly an idealization, but it is important to study for two reasons. First, the model may reveal the basic interactions among replenishment quantities at the different echelons. Second, we could choose, where possible, the pragmatic route of developing replenishment strategies based on deterministic demand, and then, conditional on these results, establishing safety stocks to provide appropriate protection against uncertainties.

If there are delays in moving between echelons, the delays are constant and not a function of lot size. No stockouts are permitted in the system.

Let us introduce some preliminary notation. Let Q_{r} and Q_{w} denote the variables of the problem, which correspond to the order quantity at the retailer (in units) and the order quantity at the warehouse (in units), respectively. In addition, we present below the parameters of the model.

D	Constant deterministic demand rate, in units/year.
A_{r}	Fixed ordering cost of a replenishment at the retailer, in money units.
A_{w}	Fixed ordering cost of a replenishment at the warehouse in money units.
$\alpha(Q_{\mathrm{r}})$	Average number of damaged items per shipment from the warehouse to the retailer, when the order quantity at the retailer is Q_{r}.
h_{r}	Inventory holding cost rate at the retailer, in money/unit/year.
h_{w}	Inventory holding cost rate at the warehouse, in money/unit/year.
J_{r}	Inventory capacity at the retailer, in units.
V	Vehicle capacity, in units.
Q_0	Maximum quantity to order at the retailer, in units (i.e. $\min\{J_{\mathrm{r}}, V, D\}$)
HOC	Sum of the total holding plus ordering costs per year.
DI	Total number of damaged items per year.

We can consider that $h_{\mathrm{w}} < h_{\mathrm{r}}$. This assumption is normal and logical, because a warehouse is generally more specialized than a retailer in terms of facilities and personnel. The average annual cost h_{w} to carry a unit of inventory at the warehouse is thus less than h_{r}, the corresponding cost at the retailer. This assumption remains throughout the rest of our article.

The goal is the minimization of the criteria (HOC, DI) so that all the demand is satisfied and no backorders occur. Two general criteria are considered. The first objective (HOC) represents the sum of costs, which are assumed to depend upon the echelon (warehouse or retailer), there being a fixed charge for ordering, and a linear per unit installation inventory holding cost. The second one (DI) represents the annual total number of damaged items which depends on the order quantity at the retailer and on the number of shipments from the warehouse to the retailer.

The two controllable (or decision) variables are the replenishment sizes Q_{r} and Q_{w}. We have to take into account that the optimality of the integer-ratio policy (the lot size at a given echelon is an integral multiple of the lot size at its successor echelon) was proved for two-echelon systems [4,11]. Therefore, we follow this integer-ratio policy and set

$$Q_{\mathrm{w}} = nQ_{\mathrm{r}}$$

where n is a positive integer.

The HOC objective corresponds to the annual inventory cost which depends on the two decision variables Q_{r} and n. Thus, considering the above integer-ratio policy this cost will be

$$HOC(Q_{\mathrm{r}}, n) = \frac{A_{\mathrm{r}}D}{Q_{\mathrm{r}}} + \frac{A_{\mathrm{w}}D}{nQ_{\mathrm{r}}} + \frac{h_{\mathrm{r}}Q_{\mathrm{r}}}{2} + h_{\mathrm{w}}\frac{(n-1)Q_{\mathrm{r}}}{2}. \tag{1}$$

The DI objective is a function of the variable Q_{r}, i.e.

$$DI(Q_{\mathrm{r}}) = \alpha(Q_{\mathrm{r}})\frac{D}{Q_{\mathrm{r}}}. \tag{2}$$

Obviously, $0 \le \alpha(Q_{\mathrm{r}}) \le Q_{\mathrm{r}} \le D$, and it seems reasonable to think that as Q_{r} increases $\alpha(Q_{\mathrm{r}})$ is non-decreasing. Besides, we assume that the average number of damaged items per shipment $\alpha(Q_r)$ is slightly sensitive to changes of the order quantity Q_r. In other words, the elasticity of $\alpha(Q_r)$ with respect to Q_r is strictly smaller than one. Therefore, we admit that $\alpha(Q_r)$ is a non-decreasing function on $[0, D]$, with $\alpha(0) = 0$ and $\alpha'(Q_r)Q_r < \alpha(Q_r)$. Observe that if the elasticity is equal to one, then $\alpha(Q_r) = kQ_r$ for some $k \in [0, 1]$, and $DI(Q_r) = kD$. Hence, it would not make sense to consider the second criterion $DI(Q_r)$, since it remains constant.

There are two constraints for the problem. The first concerns the maximum quantity to order at the retailer, which depends on the minimum between the inventory capacity at the retailer and the capacity of the vehicle for delivery. The second constraint restricts n to be a positive integer. Thus, the problem consists of finding Q_r and an integer n that minimize (1) and (2), subject to $0 < Q_r \le Q_0$.

It is worth noting that this problem is a two-objective non-linear mixed-integer programming problem. Unfortunately, these kinds of problems are not easy to solve. Continuous multiobjective problems can be studied using scalarization results or constrained parametric optimization, which most of the times is a tedious task. Integer or combinatorial multiobjective problems are very complex enumeration problems that can be faced using techniques such as dynamic programming, branch-and-bound and branch-and-cut, among others, to obtain a formal approach to the optimal solution set. Non-linear mixed-integer multiobjective problems have all the difficulties inherent in the two former families of problems. In fact, it is not possible to use any of the tools valid either for the continuous or the discrete multiobjective problems and therefore, it is necessary to develop specific approaches for each new problem. In spite of their difficulty, it is possible to find an appropriate way to solve the considered model, performing a complete case analysis of the problem and identifying the whole set of non-dominated solutions. These results are presented in the following sections.

3. Preliminary results

As we commented previously, our problem fits into a two-objective non-linear mixed-integer programming model for the warehouse/retailer system under deterministic demand. To deal with this problem it is appropriate to use the multiple criteria decision making (MCDM) methodology. The goal is to find the non-dominated solution set, also called Pareto-optimal solution set, of the bicriteria biechelon inventory/distribution (BBID) problem given by:

$$\begin{array}{lll} \text{BBID:} & v - \min(HOC(Q_r, n), DI(Q_r)) & \\ & \text{s.t.} \quad Q_r \in (0, Q_0] & \\ & \qquad\;\; n \text{ positive integer.} & \end{array} \tag{3}$$

Thus, the Pareto-optimal solution set P is defined as

$$P = \{(Q_r, n) |\text{ there does not exist } (Q_r', n') : HOC(Q_r', n') \le HOC(Q_r, n) \text{ and } DI(Q_r') \le DI(Q_r), \\ \text{with at least one of the inequalities being strict}\}.$$

Before characterizing the set P, some specific properties of the objective functions are stated. First, it can be easily shown that $DI(Q_r)$ is a strictly decreasing function on $(0, D]$.

Additionally, it is clear that function $HOC(Q_r, n)$ is convex in the region: $K = \{(Q_r, n): 0 < n < \infty, 0 < Q_r \leq B(n)\}$, where

$$B(n) = \frac{1}{n}\sqrt{\frac{2A_wD}{h_w}\left[2\sqrt{1+\frac{A_r}{A_w}n} - 1\right]}, \tag{4}$$

and $HOC(Q_r, n)$ reaches its global minimum at (Q_r^*, n^*), being

$$Q_r^* = \sqrt{\frac{2A_rD}{h_r - h_w}}, \tag{5}$$

$$n^* = \sqrt{\frac{(h_r - h_w)A_w}{h_wA_r}}. \tag{6}$$

Furthermore, for a fixed n, the value of Q_r which minimizes function $HOC(Q_r, n)$ is given by

$$\overline{Q}_r(n) = \sqrt{\frac{2D(A_rn + A_w)}{n^2h_w + n(h_r - h_w)}}. \tag{7}$$

On the contrary, when Q_r is fixed and n is considered as a real-valued variable, the value of n which makes function $HOC(Q_r, n)$ minimal can be obtained by

$$\widehat{n}(Q_r) = \frac{1}{Q_r}\sqrt{\frac{2A_wD}{h_w}} \tag{8}$$

Assuming that n and Q_r are real-valued variables, both $\overline{Q}_r(n)$ and $\widehat{n}(Q_r)$ are strictly decreasing convex functions of n and Q_r, respectively. This statement is easily proved since the first derivatives of both functions exist and they are increasing with respect to n and Q_r, respectively. In order to show when function $\overline{Q}_r(n)$ is greater than $\widehat{n}(Q_r)$ or vice-versa, let us define the following expression derived from (8),

$$\widehat{Q}_r(n) = \frac{1}{n}\sqrt{\frac{2A_wD}{h_w}}. \tag{9}$$

Moreover, it can be easily shown that functions $\overline{Q}_r(n)$ and $\widehat{Q}_r(n)$ intercept at point (Q_r^*, n^*), given by (5) and (6).

LEMMA 1 *If* $n \geq n^*$, *then* $\overline{Q}_r(n)$ *is greater than or equal to* $\widehat{Q}_r(n)$, *and the reverse when* $n < n^*$.

Proof If $n \geq n^*$, then $n^2 \geq ((h_r - h_w)A_w)/h_wA_r$ and, hence $2Dh_wA_rn^2 \geq 2D(h_r - h_w)A_w$. Thus, adding $2DA_wnh_w$ and multiplying by n both terms of the previous expression, we obtain that

$$\frac{2D(A_rn + A_w)}{n^2h_w + n(h_r - h_w)} \geq \frac{2A_wD}{n^2h_w}$$

or, in other words, $\overline{Q}_r(n) \geq \widehat{Q}_r(n)$ (see Figure 1).

Otherwise, if $n < n^*$, it is clear that $\overline{Q}_r(n) < \widehat{Q}_r(n)$. ■

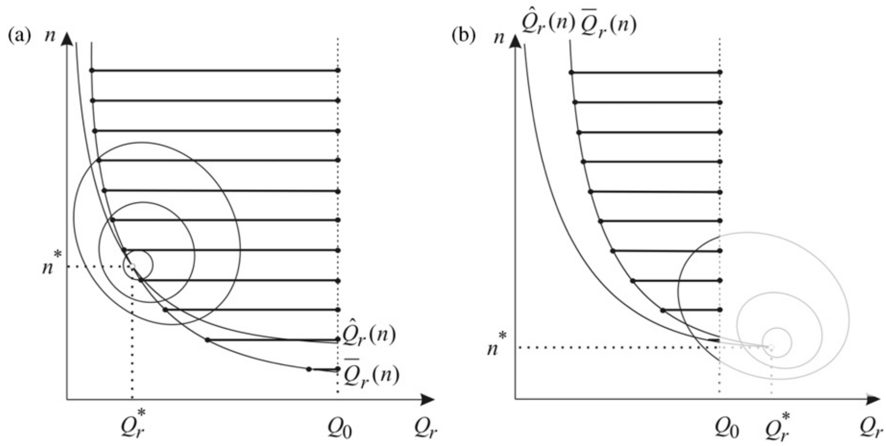

Figure 1. Illustration of $\overline{Q}_{\mathrm{r}}(n)$, $\widehat{Q}_{\mathrm{r}}(n)$ and some level curves in $\mathcal{F}$. (a) situation when $Q_0 \leq Q_{\mathrm{r}}^*$, and (b) when $Q_0 > Q_{\mathrm{r}}^*$.

LEMMA 2 *For a fixed* n, $n > 1$, *the functions* $HOC(Q_{\mathrm{r}}, n)$ *and* $HOC(Q_{\mathrm{r}}, n-j)$, $1 \leq j \leq n-1$, *intercept in a unique value* $Q_{\mathrm{r}}^{n,n-j}$, *which is given by*

$$Q_{\mathrm{r}}^{n,n-j} = \sqrt{\frac{2A_{\mathrm{w}}D}{h_{\mathrm{w}}n(n-j)}}. \tag{10}$$

Besides, $\widehat{Q}_{\mathrm{r}}(n) < Q_{\mathrm{r}}^{n,n-j} < \widehat{Q}_{\mathrm{r}}(n-j)$.

Proof Just computing $HOC(Q_{\mathrm{r}}, n) = HOC(Q_{\mathrm{r}}, n-j)$ and, taking into account that

$$\widehat{Q}_{\mathrm{r}}(n) = \sqrt{\frac{2A_{\mathrm{w}}D}{n^2 h_{\mathrm{w}}}} < Q_{\mathrm{r}}^{n,n-j} \quad \text{and} \quad \widehat{Q}_{\mathrm{r}}(n-j) = \sqrt{\frac{2A_{\mathrm{w}}D}{(n-j)^2 h_{\mathrm{w}}}} > Q_{\mathrm{r}}^{n,n-j},$$

the result follows. ■

Characterizing the non-dominated solution set P requires to consider the level curves of function $HOC(Q_{\mathrm{r}}, n)$. Accordingly, let us denote the family $\mathcal{F}$ of level curves by

$$\mathcal{F} = \left\{\varphi_l(Q_{\mathrm{r}}, n) = 0\colon \varphi_l(Q_{\mathrm{r}}, n) = [h_{\mathrm{r}} + h_{\mathrm{w}}(n-1)]nQ_{\mathrm{r}}^2 - 2\ln Q_{\mathrm{r}} + 2D(A_{\mathrm{w}} + A_{\mathrm{r}}n),\ l \in \mathbb{R} \setminus \{0\}\right\}$$

Notice that these curves are the level curves of $HOC(Q_{\mathrm{r}}, n)$, i.e. they are sets of the form $\{(Q_{\mathrm{r}}, n) \in \mathbb{R}^2 : HOC(Q_{\mathrm{r}}, n) = 1\}$. Since $HOC(Q_{\mathrm{r}}, n)$ is convex in K, the region bounded by curve $\varphi_l(Q_{\mathrm{r}}, n) = 0$ corresponds to a convex set for any value l (see Figure 1).

4. Characterization of the Pareto set

We start this section discarding those points in $\mathbb{R}^2$ which are not to be included in P with certainty. The following lemmas reduce the admissible set of candidate points to be Pareto solutions to those that belong to a given region.

LEMMA 3 *The non-dominated solution set P is included inside region R, characterized by*

$$R = \left\{(Q_r, n): \overline{Q}_r(n) \le Q_r \le Q_0, \quad \text{and } n \text{ is a positive integer}\right\} \tag{11}$$

Proof By contradiction, let us assume that point (Q_r, n) is a feasible solution which is not in R, i.e. $Q_r < \overline{Q}_r(n)$. Then (Q_r, n) is dominated by $(\overline{Q}_r(n), n)$ since both criteria would be improved by convexity of HOC and because DI is strictly decreasing with Q_r. ■

Since the characterization of the Pareto solution set depends on the relative positions of Q_r* and Q_0, we should distinguish two possible cases, namely, if $Q_0 \le Q_r$* or the reverse. The candidate points to be Pareto-optimal solutions are plotted as bold lines in Figure 1.

4.1. *When* $Q_0 \le Q_r^*$

We need to introduce the following notation before characterizing the non-dominated solution set. Let $\widehat{n}_0$ denote the integer value of $\widehat{n}(Q_0)$ where function $HOC(Q_0, n)$ reaches its minimum, i.e. $\widehat{n}_0 = \arg\{\min_{n \in \{\lfloor \widehat{n}(Q_0) \rfloor, \lceil \widehat{n}(Q_0) \rceil\}} HOC(Q_0, n)\}$, where $\lfloor \widehat{n}(Q_0) \rfloor$ and $\lceil \widehat{n}(Q_0) \rceil$ represent the largest integer smaller than and the smallest integer larger than $\widehat{n}(Q_0)$, respectively. In case of $HOC(Q_0, \lfloor \widehat{n}(Q_0) \rfloor) = HOC(Q_0, \lceil \widehat{n}(Q_0) \rceil)$, then we set $\widehat{n}_0 = \lfloor \widehat{n}(Q_0) \rfloor$. Furthermore, assuming that $\bar{n}(Q_0)$ stands for the value such that $\overline{Q}_r(n_0) = Q_0$, let $\bar{n}_0$ be the closest integer value greater than $\bar{n}(Q_0)$. Since $Q_0 \le Q_r^*$, by virtue of Lemma 1, it is clear that $\bar{n}_0 \ge \lceil \widehat{n}(Q_0) \rceil$ and hence $\bar{n}_0 \ge \widehat{n}_0$.

LEMMA 4 *Those points* (Q_r, n) *in R with* $n > \bar{n}_0$ *or* $n < \widehat{n}_0$ *are not included in P.*

Proof By contradiction, we assume that (Q_r, n) is an efficient point with $n > \bar{n}_0$. Let (Q_0, n_1) be the point where the straight line joining points (Q_r, n) and (Q_r^*, n^*) intercepts with line $Q_r = Q_0$ (see Figure 2(a)). Since function HOC is convex, $HOC(Q_0, n_1)$ is smaller than $HOC(Q_r, n)$ and, also, $DI(Q_r) > DI(Q_0)$ because $Q_r < Q_0$. Therefore, (Q_r, n) is dominated by (Q_0, n_1). Moreover, by Lemma 1 and by convexity of function $\widehat{Q}_r(n)$, point (Q_0, n_1) is even dominated by $(Q_0, \bar{n}_0)$. Therefore, point (Q_r, n) cannot be an efficient point.

Following a similar reasoning, it can be shown that any point (Q_r, n) with $n < \widehat{n}_0$ is dominated by point $(Q_0, \widehat{n}_0)$. ■

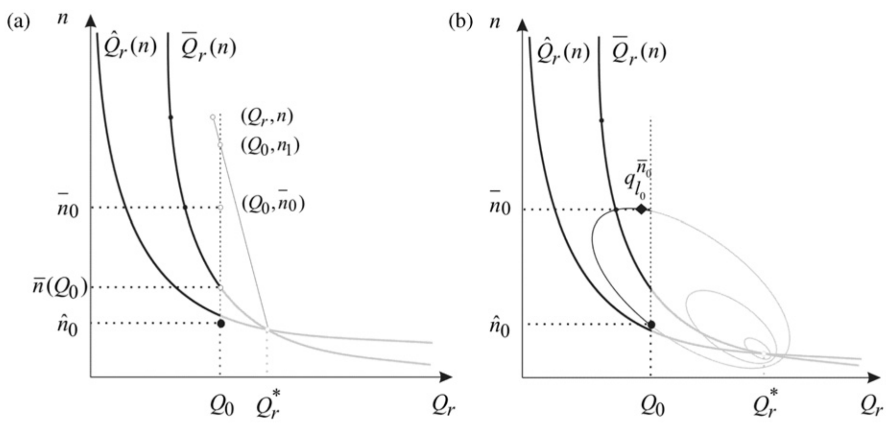

Figure 2. (a) Illustration of Lemma 4, and (b) Illustration of Theorem 5 when $\bar{n}_0 = \widehat{n}_0 + 1$.

We can now use the level curves of function *HOC* introduced before to simplify the characterization of set *P*. Accordingly, let q_l^i denote the greatest value of Q_{r} where curve $\varphi_l(Q_{\mathrm{r}}, n) = 0$ intercepts with line $n = i$. In particular, let $q_{l_0}^{\bar{n}_0}$ be the greatest value, if it exists, on the straight line $n = \bar{n}_0$ with $l_0 = HOC(Q_0, \widehat{n}_0)$, i.e. the greatest value such that $HOC(q_{l_0}^{\bar{n}_0}, \bar{n}_0) = HOC(Q_0, \widehat{n}_0)$. Additionally, let $q_{l_0}^{\widehat{n_0+1}}$ denote the greatest value, if it exists, on the straight line $n = \widehat{n}_0 + 1$ such that $HOC(q_{l_0}^{\widehat{n_0+1}}, \widehat{n}_0 + 1) = HOC(Q_0, \widehat{n}_0)$. The following theorem uses these values to identify the non-dominated solution set in the case of $Q_0 \leq Q_{\mathrm{r}}^*$.

THEOREM 5 *When $Q_0 \leq Q_{\mathrm{r}}^*$, the Pareto solution set P, assuming that $l_0 = HOC(Q_0, \widehat{n}_0)$, is given as follows*

(1) *if* $\bar{n}_0 = \widehat{n}_0$, $\quad : P = \{(Q_{\mathrm{r}}, \bar{n}_0) : Q_{\mathrm{r}} \in [\overline{Q}_{\mathrm{r}}(\bar{n}_0), Q_0]\}$

(2) *if* $\bar{n}_0 = \widehat{n}_0 + 1$,

- (a) *if* $\overline{Q}_{\mathrm{r}}(\bar{n}_0) \leq q_{l_0}^{\bar{n}_0} \leq Q_0$ $\quad : P = \{(Q_{\mathrm{r}}, \bar{n}_0) : Q_{\mathrm{r}} \in [\overline{Q}_{\mathrm{r}}(\bar{n}_0), q_{l_0}^{\bar{n}_0})\} \cup \{(Q_0, \widehat{n}_0)\}$
- (b) *if* $q_{l_0}^{\bar{n}_0} > Q_0$ $\quad : P = \{(Q_{\mathrm{r}}, \bar{n}_0) : Q_{\mathrm{r}} \in [\overline{Q}_{\mathrm{r}}(\bar{n}_0), Q_0]\}$
- (c) *otherwise* $\quad : P = \{(Q_0, \widehat{n}_0)\}$

(3) *if* $\bar{n}_0 > \widehat{n}_0 + 1$,

- (a) *if* $q_{l_0}^{\widehat{n_0+1}} = Q_0$ $\quad : P = \{(Q_0, \widehat{n}_0 + 1), (Q_0, \widehat{n}_0)\}$
- (b) *otherwise* $\quad : P = \{(Q_0, \widehat{n}_0)\}$

Proof By virtue of Lemma 4, the candidate points to be Pareto solutions are of the form (Q_{r}, n) with $\widehat{n}_0 \leq n \leq \bar{n}_0$. In particular, when $\bar{n}_0 = \widehat{n}_0$ or $\bar{n}_0 = \widehat{n}_0 + 1$, these points lie on the line $n = \bar{n}_0$ from $\overline{Q}_{\mathrm{r}}(\bar{n}_0)$ to the value corresponding to $\min\{q_{l_0}^{\bar{n}_0}, Q_0\}$ and, also point $(Q_0, \widehat{n}_0)$, which is represented by the largest black dot in Figure 2(a) and (b). Moreover, the fact that $q_{l_0}^{\bar{n}_0}$ does not exist implies that there is no point on $n = \bar{n}_0$ with *HOC* value equal to $l_0 = HOC(Q_0, \widehat{n}_0)$. Indeed, this result indicates that the *HOC* value of any point on line $n = \bar{n}_0$ is greater than $HOC(Q_0, \widehat{n}_0)$, since any point on this line can be seen as the interception point between $\varphi_l(Q_{\mathrm{r}}, n) = 0$ and $n = \bar{n}_0$, with $l > HOC(Q_0, \widehat{n}_0)$. Therefore, when either $q_{l_0}^{\bar{n}_0}$ does not exist or $q_{l_0}^{\bar{n}_0} = Q_0$, the Pareto solution set is of the form: $P = \{(Q_0, \widehat{n}_0)\}$. On the contrary, if $q_{l_0}^{\bar{n}_0}$ exists, two cases can arise, namely, $\overline{Q}_{\mathrm{r}}(\bar{n}_0) \leq q_{l_0}^{\bar{n}_0} \leq Q_0$ or $Q_0 < q_{l_0}^{\bar{n}_0}$. Notice that, by Lemma 3, the case $\overline{Q}_{\mathrm{r}}(\bar{n}_0) > q_{l_0}^{\bar{n}_0}$ leads to dominated solutions.

Thus, when $\overline{Q}_{\mathrm{r}}(\bar{n}_0) \leq q_{l_0}^{\bar{n}_0} \leq Q_0$, there exists a point $(q_{l_0}^{\bar{n}_0}, \bar{n}_0)$, depicted as a rhomb in Figure 2(b), with the same value of *HOC* than point $(Q_0, \widehat{n}_0)$ but with worse value for the second criterion. Hence, $(q_{l_0}^{\bar{n}_0}, \bar{n}_0)$ is dominated by $(Q_0, \widehat{n}_0)$. In addition, since function $\overline{Q}_{\mathrm{r}}(n)$ reaches its minimum at $(\overline{Q}_{\mathrm{r}}(\bar{n}_0), \bar{n}_0)$ when $n = \bar{n}_0$, all the points in $[\overline{Q}_{\mathrm{r}}(\bar{n}_0), q_{l_0}^{\bar{n}_0})$ have smaller *HOC* value than point $(q_{l_0}^{\bar{n}_0}, \bar{n}_0)$ and, therefore, they are non-dominated solutions. Accordingly, the Pareto solution set is as follows: $P = \{(Q_{\mathrm{r}}, \bar{n}_0) : Q_{\mathrm{r}} \in [\overline{Q}_{\mathrm{r}}(\bar{n}_0), q_{l_0}^{\bar{n}_0})\} \cup (Q_0, \widehat{n}_0)$.

On the contrary, if $\overline{Q}_{\mathrm{r}}(\bar{n}_0) < Q_0 < q_{l_0}^{\bar{n}_0}$, we can also exploit the fact that $\overline{Q}_{\mathrm{r}}(n)$ is strictly convex to guarantee that points $[\overline{Q}_{\mathrm{r}}(\bar{n}_0), Q_0]$ on line $n = \bar{n}_0$ have smaller *HOC* values than $(Q_0, \widehat{n}_0)$ and, hence, $P = \{(Q_{\mathrm{r}}, \bar{n}_0)$: $Q_{\mathrm{r}} \in [\overline{Q}_{\mathrm{r}}(\bar{n}_0), Q_0]\}$.

When $\bar{n}_0 > \widehat{n}_0 + 1$, the unique non-dominated solution is point $(Q_0, \widehat{n}_0)$ unless $q_{l_0}^{\widehat{n_0+1}} = Q_0$, in such a case, the Pareto solution set contains points $(Q_0, \widehat{n}_0 + 1)$ and $(Q_0, \widehat{n}_0)$. ■

4.2. When $Q_0 > Q_r^*$

From now on, let $\bar{n}$ denote the closest integer value to n^* which minimizes $HOC(\overline{Q}_r(n), n)$, that is, $\bar{n} = \arg\{\min_{n\in\{\lfloor n^*\rfloor, \lceil n^*\rceil\}} HOC(\overline{Q}_r(n), n)\}$, with $\overline{Q}_r(n) \leq Q_0$, and where $\lfloor n^*\rfloor$ stands for the closest integer value smaller than n^*, and $\lceil n^*\rceil$ is the closest integer value greater than n^*. In case of $HOC(\overline{Q}_r(\lfloor n^*\rfloor), \lfloor n^*\rfloor) = HOC(\overline{Q}_r(\lceil n^*\rceil), \lceil n^*\rceil)$, we set $\bar{n} = \lfloor n^*\rfloor$ since, by convexity of $\overline{Q}_r(n)$, point $(\overline{Q}_r(\lfloor n^*\rfloor), \lfloor n^*\rfloor)$ is to the right of $(\overline{Q}_r(\lceil n^*\rceil), \lceil n^*\rceil)$ and, therefore, the second criterion $DI(Q_r)$ is improved. Observe that, from definition of $\bar{n}$, $\bar{n} \geq \widehat{n}_0$ when $Q_0 > Q_r^*$.

The admissible set of candidate points to be Pareto solutions can be more specifically characterized, according to the following lemmas.

Lemma 6 *When $Q_0 > Q_r^*$, those points (Q_r, n) in R with $n < \widehat{n}_0$ or $n > \bar{n}$ are not to be included in P.*

Proof Let (Q_r, n) be an efficient point with $n > \bar{n}$. To show, by contradiction, that (Q_r, n) cannot be a non-dominated point we should distinguish two cases, namely, when $Q_r \leq Q_r^*$ and $Q_r > Q_r^*$. We first focus our attention on the case $Q_r \leq Q_r^*$. Accordingly, let p_1 denote point (Q_r, n). Since function HOC is strictly convex, point p_1 is dominated by that point corresponding to the interception point (plotted by a white dot in Figure 3) of the segment line joining points p_1 and (Q_r^*, n^*) with straight line $n = \bar{n}$. On the other hand, when $Q_r > Q_r^*$, let p_2 denote point (Q_r, n). In this case, since function $\widehat{Q}_r(n)$ provides the point with minimum HOC cost for a fixed Q_r, it is easy to see that point p_2 is dominated by point $(Q_r, \lceil\widehat{n}(Q_r)\rceil)$ depicted by a white dot in Figure 3. Therefore, in both cases, point (Q_r, n) is dominated.

Moreover, applying the same reasoning, namely, that function $\widehat{Q}_r(n)$ provides the point with minimum HOC cost for a fixed Q_r, it can be easily shown that any point $p_3 = (Q_r, n)$ with $n < \widehat{n}_0$ is dominated by point $(Q_r, \widehat{n}_0)$. ■

As a result of Lemma 6, the maximum number of intervals containing non-dominated solutions is $k = \bar{n} - \widehat{n}_0 + 1$.

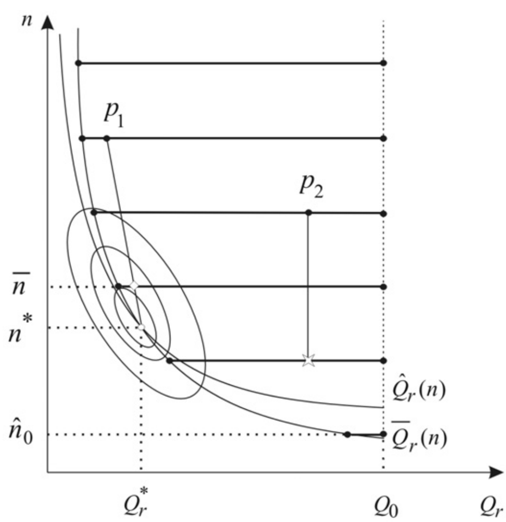

Figure 3. Illustration of Lemma 6.

We show below that the Pareto solution set P consists of union of intervals, which are located in different lines n, with $\widehat{n}_0 \le n \le \bar{n}$. In what follows, we denote by $P(n)$ the set of non-dominated points on line n. Therefore, the Pareto solution set is given by $P = \cup_{n=\widehat{n}_0}^{\bar{n}} P(n)$. First, we need to show that $HOC(\overline{Q}_{\mathrm{r}}(\bar{n}), \bar{n})$, $HOC(\overline{Q}_{\mathrm{r}}(\bar{n}-1), \bar{n}-1), \ldots,$ $HOC(\overline{Q}_{\mathrm{r}}(\widehat{n}_0), \widehat{n}_0)$ represent a sequence of increasing values.

Proposition 7 *For all n with $\widehat{n}_0 < n \le \bar{n}$, $HOC(\overline{Q}_{\mathrm{r}}(n), n) < HOC(\overline{Q}_{\mathrm{r}}(n-1), n-1)$ holds.*

Proof Without loss of generality, consider values n^*, $\bar{n}$ and $\bar{n}-1$. By contradiction, let us assume that $HOC(\overline{Q}_{\mathrm{r}}(\bar{n}), \bar{n}) \ge HOC(\overline{Q}_{\mathrm{r}}(\bar{n}-1), \bar{n}-1)$. Since (Q_{r}^*, n^*) represents the point where function HOC reaches the minimum, we have that $HOC(\overline{Q}_{\mathrm{r}}(n^*) = Q_{\mathrm{r}}^*, n^*) < HOC(\overline{Q}_{\mathrm{r}}(\bar{n}), \bar{n})$. Therefore, there should be two points $a = (Q^1, \bar{n})$ and $b = (Q^2, \bar{n})$ on $n = \bar{n}$, with $Q^1 < Q^2$, which intercept with curve $\varphi_l(Q_{\mathrm{r}}, n) = 0$, where $l = HOC(\overline{Q}_{\mathrm{r}}(\bar{n}-1), \bar{n}-1)$ (see Figure 4). Accordingly, the HOC value in points a and b on $n = \bar{n}$ coincides with $HOC(\overline{Q}_{\mathrm{r}}(\bar{n}-1), \bar{n}-1)$, so points a, b and $(\overline{Q}_{\mathrm{r}}(\bar{n}-1), \bar{n}-1)$ are included in the same level curve $\varphi_l(Q_{\mathrm{r}}, n) = 0$. However, this result contradicts the fact that $(\overline{Q}_{\mathrm{r}}(\bar{n}), \bar{n})$ is the unique point that minimizes function HOC for $n = \bar{n}$, and hence, inequality $HOC(\overline{Q}_{\mathrm{r}}(\bar{n}), \bar{n}) \ge HOC(\overline{Q}_{\mathrm{r}}(\bar{n}-1), \bar{n}-1)$ is not possible. ∎

We show later on that in the determination of the whole Pareto set it is only necessary to successively determine the non-dominated solutions corresponding to consecutive values of n. Thus, taking into account that $\overline{Q}_{\mathrm{r}}(m) < \overline{Q}_{\mathrm{r}}(m-1)$ for a fixed integer value m, since $\overline{Q}_{\mathrm{r}}(n)$ is a strictly decreasing function, and according to Proposition 7, the only two combinations of HOC values for consecutive values of n are depicted in Figure 5. Therefore, it is clear that the Pareto set is updated adding a new interval on line $n = m-1$, which starts from point $\max\{Q_{\mathrm{r}}^{m,m-1}, \overline{Q}_{\mathrm{r}}(m-1)\}$ for $\widehat{n}_0 < m \le \bar{n}$, and $Q_{\mathrm{r}}^{m,m-1}$ given by (10). The following corollary sheds light on the determination of efficient solutions when only two consecutive values of n are considered.

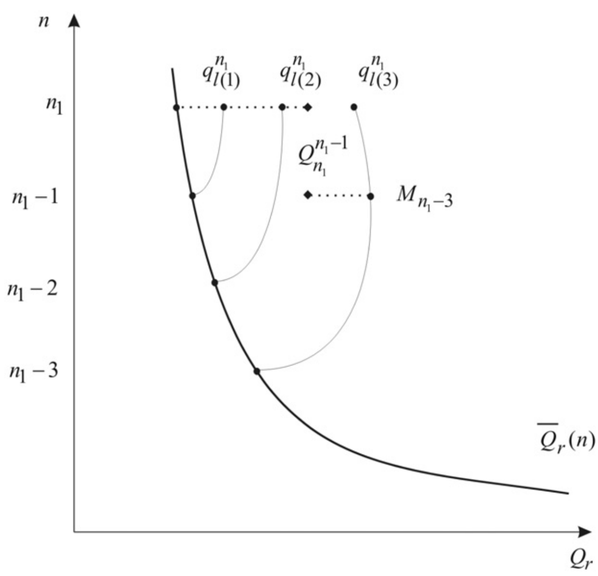

Figure 4. Illustration of Proposition 7.

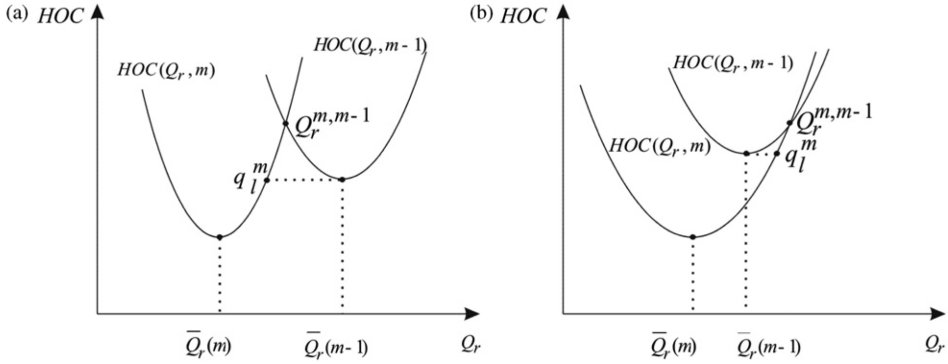

Figure 5. Feasible cases when two *HOC* functions corresponding to two consecutive values of n are faced.

Corollary 8 *Given lines $n = m$ and $n = m - 1$, the sets of non-dominated solutions $P(m)$ and $P(m-1)$ on these lines are given as follows*:

(1) *If* $\overline{Q}_{\mathrm{r}}(m-1) = \max\{Q_{\mathrm{r}}^{m,m-1}, \overline{Q}_{\mathrm{r}}(m-1)\}$ *then*

$$P(m) = [\overline{Q}_{\mathrm{r}}(m), q_l^m),\ \textit{with}\ l = HOC(\overline{Q}_{\mathrm{r}}(m-1), m-1),\ \textit{and}\ P(m-1) = [\overline{Q}_{\mathrm{r}}(m-1), a_1]$$

(2) *If* $Q_{\mathrm{r}}^{m,m-1} = \max\{Q_{\mathrm{r}}^{m,m-1}, \overline{Q}_{\mathrm{r}}(m-1)\}$ *then*

$$P(m) = [\overline{Q}_{\mathrm{r}}(m), Q_{\mathrm{r}}^{m,m-1}),\ \textit{and}\ P(m-1) = [Q_{\mathrm{r}}^{m,m-1}, b_1]$$

where values a_1 and b_1 depend on the interception points with curve $HOC(Q_{\mathrm{r}}, m-2)$.

The following result shows that the determination of P is reduced to successively evaluate functions *HOC* corresponding to two consecutive values of n.

Theorem 9 *The Pareto-optimal solution set can be computed via pairwise comparison of functions HOC corresponding to consecutive values of n.*

Proof Assume that the efficient points related to lines $n = m$ and $n = m - 1$ have been already determined. Accordingly, the sets $P(m)$ and $P(m-1)$ are obtained considering only one of the two cases, namely (1) and (2), introduced in Corollary 8. Moreover, consider that we analyse line $n = m - 2$ in the process of determination of P. Likewise, the set $P(m-2)$ is given by either case (1) or case (2) in Corollary 8 as a result of the comparison of the curves $HOC(Q_{\mathrm{r}}, m-1)$ and $HOC(Q_{\mathrm{r}}, m-2)$. Hence, four possible relationships between the pairs of curves $HOC(Q_{\mathrm{r}}, m)$, $HOC(Q_{\mathrm{r}}, m-1)$ and $HOC(Q_{\mathrm{r}}, m-1)$, $HOC(Q_{\mathrm{r}}, m-2)$ can arise, namely 1–1, 1–2, 2–1, 2–2 (see Figure 6). We proceed to evaluate each combination separately.

(a) Case 1–1: When the pairs of curves $HOC(Q_{\mathrm{r}}, m)$, $HOC(Q_{\mathrm{r}}, m-1)$ and $HOC(Q_{\mathrm{r}}, m-1)$, $HOC(Q_{\mathrm{r}}, m-2)$ are both of the form (1), it is easily proved that $P(m) = [\overline{Q}_{\mathrm{r}}(m), q_l^m)$, with $l = HOC(\overline{Q}_{\mathrm{r}}(m-1), m-1)$, $P(m-1) = [\overline{Q}_{\mathrm{r}}(m-1), q_{l'}^{m-1}]$, with $l' = HOC(\overline{Q}_{\mathrm{r}}(m-2), m-2)$ and the interval associated to $P(m-2)$ begins at point $\overline{Q}_{\mathrm{r}}(m-2)$ (see Figure 6(a)). Therefore, case 1–1 is reduced to analyse two cases of the form (1) independently since adding curve

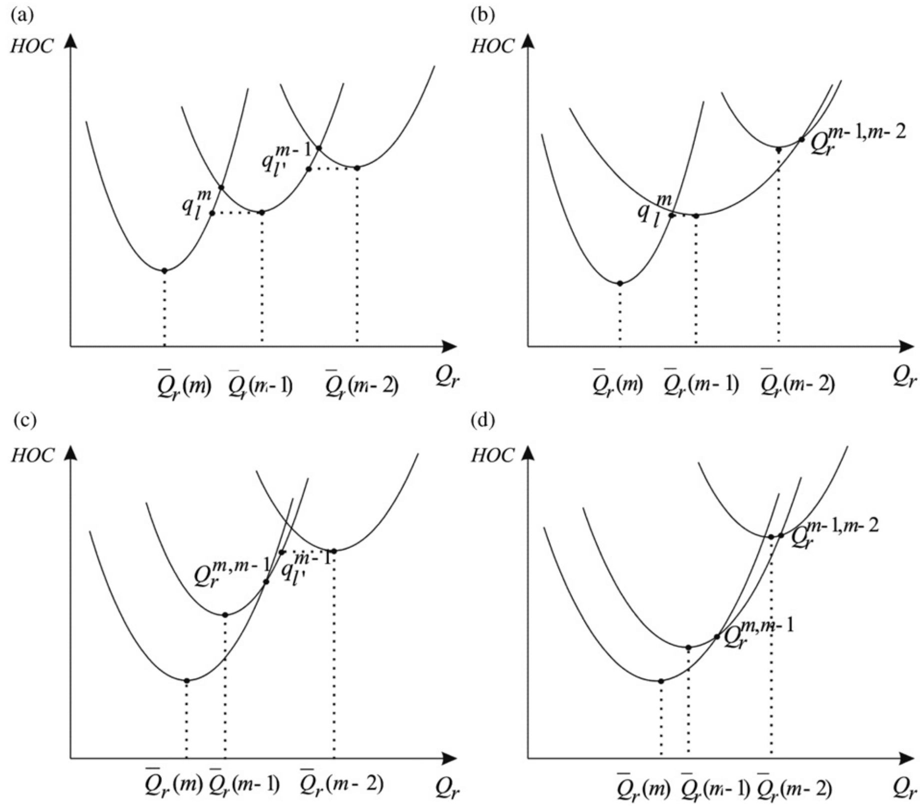

Figure 6. Admissible cases when three consecutive *HOC* functions are compared.

$HOC(Q_r, m-2)$ does not alter the efficient solutions corresponding to curves $HOC(Q_r, m)$ and $HOC(Q_r, m-1)$.

(b) Case 1–2: When the combination of curves $HOC(Q_r, m)$ and $HOC(Q_r, m-1)$ is of the form 1 and the pair of curves $HOC(Q_r, m-1)$ and $HOC(Q_r, m-2)$ corresponds to the type 2, it can be easily shown that $P(m) = [\overline{Q}_r(m), q_l^m)$, with $l = HOC(\overline{Q}_r(m-1), m-1)$, $P(m-1) = [\overline{Q}_r(m-1), Q_r^{m-1,m-2}]$ and the interval associated to $P(m-2)$ begins at point $Q_r^{m-1,m-2}$ (see Figure 6b)). Again, the inclusion of curve $HOC(Q_r, m-2)$ does not affect the efficient solutions corresponding to curves $HOC(Q_r, m)$ and $HOC(Q_r, m-1)$, and hence, case 1–2 can be dealt with separately.

(c) Case 2–1: When the combination of curves $HOC(Q_r, m)$ and $HOC(Q_r, m-1)$ is of the form (1) and the pair of curves $HOC(Q_r, m-1)$ and $HOC(Q_r, m-2)$ corresponds to the type (1), two different situations can arise. In particular, we must distinguish two cases, namely, if $HOC(\overline{Q}_r(m-2), m-2) > HOC(Q_r^{m,m-1}, m-1)$ or $HOC(\overline{Q}_r(m-2), m-2) \leq HOC(Q_r^{m,m-1}, m-1)$. The latter case, depicted in Figure 7, is not feasible since $q_{l'}^{m-1} < q_{l'}^m < \overline{Q}_r(m-2)$ with $l' = HOC(\overline{Q}_r(m-2), m-2)$, which contradicts the fact that level curve $\varphi_{l'}(Q_r, m) = 0$ is a convex set containing level curves $\varphi_l(Q_r, n) = 0$, with $m < m' \leq \bar{n}$ and $l = HOC(\overline{Q}_r(m'), m')$. Thus, the unique valid alternative is that

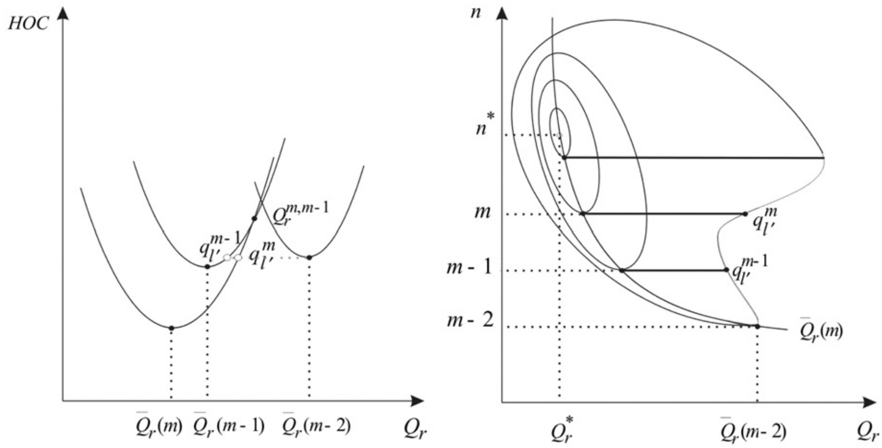

Figure 7. Unfeasible case when $HOC(\overline{Q}_\mathrm{r}(m-2), m-2) \leq HOC(Q_\mathrm{r}^{m,m-1}, m-1)$.

$HOC(\overline{Q}_\mathrm{r}(m-2), m-2) > HOC(Q_\mathrm{r}^{m,m-1}, m-1)$ (see Figure 6(c)), and hence, combination 2–1 can be analysed separately to give $P(m) = [\overline{Q}_\mathrm{r}(m), Q_\mathrm{r}^{m,m-1})$, $P(m-1) = [Q_\mathrm{r}^{m,m-1}, q_{l'}^{m-1})$ and the interval $P(m-2)$ starting from $\overline{Q}_\mathrm{r}(m-2)$.

(d) Case 2–2: When the pairs of curves $HOC(Q_\mathrm{r}, m)$, $HOC(Q_\mathrm{r}, m-1)$ and $HOC(Q_\mathrm{r}, m-1)$, $HOC(Q_\mathrm{r}, m-2)$ are both of the form 2 (see Figure 6(d)), combination 2–2 is reduced to evaluate independently two consecutive cases of type (2) to give $P(m) = [\overline{Q}_\mathrm{r}(m), Q_\mathrm{r}^{m,m-1})$, $P(m-1) = [Q_\mathrm{r}^{m,m-1}, Q_\mathrm{r}^{m-1,m-2})$ and $P(m-2)$ starting from $Q_\mathrm{r}^{m-1,m-2}$.

Concluding, any feasible combination between two pairs of consecutive curves HOC is reduced to consider each pair separately. ■

The procedure to determine the whole Pareto-optimal solution set, which is based on the previous results, is sketched in Algorithm 1.

In the next section, we use this characterization of P to show that the method proposed in Bookbinder and Chen [1] to solve their bicriteria problem is not correct.

5. Bookbinder and Chen's approach

As in this article, Bookbinder and Chen [1] likewise addressed a bicriteria two-echelon inventory/distribution system. In their problem, the first criterion coincides with our function HOC, and the second one concerns the annual transportation cost $TC(Q_\mathrm{r}) = T_\mathrm{r}D/Q_\mathrm{r}$, ($T_\mathrm{r}$ represents the fixed transportation cost per shipment). While both objectives, their cost TC and our criterion DI, are conceptually different, they are characterized by the same type of function, namely, a strictly decreasing function in Q_r. Remark that $DI(Q_\mathrm{r})$ is equal to $TC(Q_\mathrm{r})$, if $\alpha(Q_\mathrm{r}) = T_\mathrm{r}$. Therefore, it seems reasonable to think that our solution method and their approach should provide the same solution for the same instance. Nevertheless, as we show below, their approach does not always provide good solutions. In particular, when demand is assumed to be known, their solution set consists of either the point (Q_0, n^*) if $Q_0 \leq Q_\mathrm{r}^*$, or otherwise, an infinite number of points (Q_r, n), with $n = n^*$ and $Q_\mathrm{r} \in [Q_\mathrm{r}^*, Q_1]$, where $Q_1 = \min\{B(n^*), Q_0\}$.

Algorithm 1: Procedure to determine the Pareto-optimal set for problem BBID

```
Data: $D$, $A_r$, $A_w$, $h_r$, $h_w$, $Q_0$ and function $\alpha$
Determine $Q_r^*$
Calculate $\hat{n}_0$
if $Q_0 \leq Q_r^*$ then
    Calculate $\bar{n}_0$
    Determine $P$ according to Theorem 5
else
    Calculate $\bar{n}$
    $n \leftarrow \bar{n}$
    $P(n) \leftarrow \emptyset$
    $P \leftarrow \emptyset$
    $Q \leftarrow \overline{Q}_r(n)$
    while $Q < Q_0$ and $n-1 \geq \hat{n}_0$ do
        if $\overline{Q}_r(n-1) = \max\{Q_r^{n,n-1}, \overline{Q}_r(n-1)\}$ then
            $P(n) = [Q, \min\{Q_0, q_l^n\})$, with $l = HOC(\overline{Q}_r(n-1), n-1)$
            $Q = \min\{Q_0, \overline{Q}_r(n-1)\}$
        else
            $P(n) = [Q, \min\{Q_0, Q_r^{n,n-1}\})$
            $Q = \min\{Q_0, Q_r^{n,n-1}\}$
        end if
        $P \leftarrow P \cup P(n)$
        $n \leftarrow n-1$
    end while
    if $n-1 < \hat{n}_0$ then
            $P \leftarrow P \cup P(n) = [Q,\ Q_0]$
    end if
end if
return $P$
```

Moreover, when $Q_0 \leq Q_r^*$, they claim that the problem has its global minimum at (Q_0, n^*), (see their theorem on p. 710 [1]). This assertion is wrong. As our Theorem 5 states, the problem formulated in (3) does not have a unique solution. Moreover, different solutions can be reached depending on the input data.

We consider Example 1 described in Bookbinder and Chen [1], and change only the demand $D = 10\,000$ to $D = 90\,000$, leaving the same values for the rest of parameters. That is, $A_r = 30$, $A_w = 20$ (transportation unit cost)$T_r = 100$, $h_r = 1$, $h_w = 0.5$ and $Q_0 = J_r = V = 1000$. According to (5), $Q_r^* = 600\sqrt{30} = 3286.33$, then we have $Q_r^* \geq Q_0$. Following Bookbinder and Chen's method, the global minimum is achieved at (Q_0, n^*). Since $Q_0 = 1000$ and $n^* = 0.816$, their solution is given by $(Q_0,\ \lceil n^* \rceil) = (1000, 1)$ with $HOC = 5000$ and $TC = T_r D / Q_r = 9000$.

However, the solution above is not good, since we can find a new point that dominates the former. Thus, following our method, we obtain that $\bar{n}_0 = 11$ and $\hat{n}_0 = 3$. Therefore, $\bar{n}_0 > \hat{n}_0 + 1$, and according to Theorem 5, the optimal solution is ($Q_0 = 1000$, $n = 3$) with costs $HOC = 4300$ and $TC = 9000$, respectively.

Secondly, when $Q_r^* \leq Q_0$, Bookbinder and Chen pointed out (see assertion (2) of their theorem on p. 710) that the problem has non-dominated solutions (Q_r, n), with $n = n^*$ and $Q_r \in [Q_r^*, Q_1]$, where $Q_1 = \min\{B(n^*), Q_0\}$. $B(n^*)$ is given in (4) and represents an upper bound necessary to guarantee that the function $HOC(Q_r, n)$ has its global minimum at (Q_r^*, n^*). Nevertheless, as it has been shown in previous sections, non-dominated solutions are arranged at different intervals, changing the n integer value in each interval.

To show this effect, we consider the same Example 2 proposed in [1], where the parameters are given as: $A_r = 100$, $A_w = 200$, $T_r = 400$, $D = 10\,000$, $h_r = 3$, $h_w = 1$, $J_r = 1500$ and $V = 2000$. Their procedure yields the following results: $n^* = 2, Q_r^* = 1000$ and $Q_0 = 1500$. Since $Q_r^* \leq Q_0$, Bookbinder and Chen asserted that the problem has an infinite number of non-dominated solutions with $n = 2$ and $1000 \leq Q_r \leq 1500$. Also, they even showed some of these solutions in their Table 1 on p. 711. Again, the authors have been wrong, because point ($Q_r = 1500$, $n = 2$), with $HOC = 4333$ and $TC = 2667$, was proposed as non-dominated solution in that table. However, this point is dominated by ($Q_r = 1500$, $n = 1$), with $HOC = 4250$ and $TC = 2666.66$.

Therefore, the non-dominated solution set P should be determined according to Algorithm 1. First, we must calculate $\bar{n} = 2$ and $\widehat{n}_0 = 1$, hence $k = \bar{n} - \widehat{n}_0 + 1 = 2$. Moreover, $\overline{Q}_r(1) = \max\{Q_r^{2,1}, \overline{Q}_r(1)\} = 1000\sqrt{2}$, $l = HOC(\overline{Q}_r(1), 1) = 6000/\sqrt{2}$ and thus $q_l^2 = 1000\sqrt{2}$. Hence, $P(2) = [\overline{Q}_r(2), q_l^2)$ and we proceed to evaluate $n = 1$ with $Q = \overline{Q}_r(1)$.

Table 1. Thirty randomly generated instances of the BBID problem.

	A_r	A_w	h_w	h_r	Q_0	D
$P1$	4.70	1.98	3.79	5.24	90.90	256.56
$P2$	3.26	7.86	0.14	0.33	44.34	38.95
$P3$	5.89	5.57	4.04	7.42	28.13	918.46
$P4$	5.70	7.31	0.14	1.50	60.30	77.23
$P5$	7.13	5.13	0.18	1.68	64.10	19.76
$P6$	1.00	4.60	0.41	0.79	77.33	63.40
$P7$	9.05	3.86	0.14	1.12	68.65	45.32
$P8$	8.17	8.83	6.60	6.79	52.36	652.69
$P9$	3.37	8.09	0.63	1.72	6.36	4.11
$P10$	9.39	9.49	1.04	4.83	60.05	611.55
$P11$	8.07	2.21	0.01	0.06	75.25	18.22
$P12$	4.17	4.25	1.20	4.32	38.81	587.49
$P13$	1.98	5.49	0.43	0.55	99.49	92.27
$P14$	8.81	7.44	1.43	5.75	97.99	416.01
$P15$	5.91	8.33	0.22	1.31	38.02	69.45
$P16$	6.52	5.47	2.24	9.26	48.30	758.28
$P17$	8.25	5.39	2.32	5.94	17.48	97.77
$P18$	2.04	4.99	0.81	1.43	13.26	26.43
$P19$	9.08	8.42	0.18	1.93	52.90	52.47
$P20$	7.94	3.13	1.08	6.16	29.23	689.42
$P21$	1.07	7.53	0.08	0.10	62.92	43.00
$P22$	7.59	6.46	0.04	0.12	13.35	76.87
$P23$	9.90	2.12	2.75	5.73	26.70	166.27
$P24$	5.34	5.90	0.36	1.85	29.17	44.79
$P25$	9.37	6.52	0.05	1.37	37.24	97.04
$P26$	7.51	8.53	0.20	0.77	68.50	94.92
$P27$	1.05	4.98	0.26	1.67	19.85	73.75
$P28$	4.45	6.17	0.11	0.26	42.89	12.90
$P29$	1.87	6.52	0.62	1.07	39.12	97.09
$P30$	7.48	8.01	0.97	0.99	58.16	96.30

Since $n-1=0<\widehat{n}_0$, the algorithm finishes determining $P(1)=[\overline{Q}_r(1), Q_0]$, therefore, the Pareto-solution set contains two intervals, namely,

$$\begin{aligned} P &= \{(Q_r, 2) : Q_r \in [\overline{Q}_r(2), \overline{Q}_r(1))\} \cup \{(Q_r, 1) : Q_r \in [\overline{Q}_r(1), Q_0]\} \\ &= \{(Q_r, 2) : Q_r \in [1000, 1000\sqrt{2}]\} \cup \{(Q_r, 1) : Q_r \in [1000\sqrt{2}, 1500]\} \end{aligned}$$

Hence, for $Q_r \geq 1000\sqrt{2}$, all those solutions proposed by Bookbinder and Chen's method are not efficient and are dominated by points $(Q_r, n=1)$.

6. Computational results

The procedure described in Algorithm 1 was implemented in C using LEDA libraries on a HP-712/60 workstation. In order to check the efficiency of this algorithm, multiple instances were randomly generated. The input data were obtained from uniform distributions on intervals, where the minimum and maximum values were different random numbers. In Table 1, 30 instances are shown.

The Pareto-optimal solution sets for the problems in Table 1 are depicted in Table 2, where the running times have been omitted since they are negligible.

The efficiency of our procedure has been tested. This test consists of generating 1000 random points for each instance. Then, we choose those which are non-dominated by using an enumerative comparison algorithm. We compare our Pareto solution set with the non-dominated randomly generated points. For each non-dominated generated point, we have to determine whether the point is included in the Pareto-optimal solution set proposed or it is dominated by a point in that set. In all the instances, the considered points either belong to our solution set or they are dominated by points in our Pareto-optimal solution set.

7. Conclusions and final remarks

In this article, we have studied a non-linear biobjective optimization model for a two-echelon serial inventory/distribution system under deterministic demand. We have characterized the non-dominated optimal solution set and proposed an algorithm to generate it.

A similar model was studied by Bookbinder and Chen [1], but unfortunately their solution method is not correct as we have shown in a previous section. The complete analysis of the problem requires a detailed study of the model. In this analysis, the problem is a mixed-integer non-linear two-objective optimization model, where neither the tools of continuous nor discrete optimization are directly applicable. We have performed this analysis decomposing the problem and integrating the solutions obtained in each subproblem into the final solution set. Two goals have been achieved in this article: to analyse and solve a mixed-integer non-linear two-objective optimization model, and to correct the solution of a model already proposed in the literature.

Further research could be carried out to study the biechelon inventory/distribution system here addressed but considering more than two criteria. Also, it could be interesting to study the behaviour of the non-linear biobjective optimization model, here studied, on a multi-echelon serial inventory/distribution system or, in general, on multi-echelon systems.

Table 2. Pareto-optimal solution sets for the problems shown in Table 1.

Problem	Variables	Pareto-optimal solution set		
$P1$	Q_r	[25.57, 90.90]		
	n	1		
$P2$	Q_r	[34.52, 44.34]		
	n	2		
$P3$	Q_r	[28.13, 28.13]		
	n	2		
$P4$	Q_r	[36.66, 60.30]	[26.57, 36.66)	[24.60, 25.37)
	n	2	3	4
$P5$	Q_r	[23.73, 64.10]	[14.35, 23.73)	
	n	1	2	
$P6$	Q_r	[29.98, 77.33]	[18.67, 26.12)	
	n	1	2	
$P7$	Q_r	[35.34, 68.65]	[28.10, 35.34)	
	n	1	2	
$P8$	Q_r	[52.36, 52.36]		
	n	1		
$P9$	Q_r	[5.09, 6.36]		
	n	2		
$P10$	Q_r	[54.27, 60.05]		
	n	2		
$P11$	Q_r	[75.25, 75.25]		
	n	1		
$P12$	Q_r	[36.60, 38.81]		
	n	2		
$P13$	Q_r	[50.06, 99.49]		
	n	1		
$P14$	Q_r	[48.49, 97.99]	[38.10, 46.31)	
	n	1	2	
$P15$	Q_r	[30.24, 38.02]	[26.25, 29.54)	
	n	2	3	
$P16$	Q_r	[44.31, 48.30]	[34.93, 42.94)	
	n	1	2	
$P17$	Q_r	[17.48, 17.48]		
	n	1		
$P18$	Q_r	[13.26, 13.26]	[10.34, 12.21]	
	n	1	2	
$P19$	Q_r	[49.54, 52.90]	[28.60, 49.54)	[23.33, 28.60)
	n	1	2	3
$P20$	Q_r	[29.23, 29.23]		
	n	2		
$P21$	Q_r	[48.06, 62.92]		
	n	2		
$P22$	Q_r	[13.35, 13.35]		
	n	12		
$P23$	Q_r	[26.41, 26.70]		
	n	1		
$P24$	Q_r	[27.09, 29.17]	[18.33, 27.09)	
	n	1	2	
$P25$	Q_r	[37.24, 37.24]		
	n	4		
$P26$	Q_r	[63.62, 68.50]	[48.00, 63.62)	
	n	1	2	

(*continued*)

Table 2. Continued.

Problem	Variables	Pareto-optimal solution set		
*P*27	Q_r	[15.34, 19.85]	[11.88, 15.34)	[10.55, 11.88)
	n	3	4	5
*P*28	Q_r	[32.46, 42.89]		
	n	1		
*P*29	Q_r	[39.02, 39.12]	[24.27, 29.23)	
	n	1	2	
*P*30	Q_r	[54.89, 58.16]		
	n	1		

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